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Exponents and Radicals in Algebra

This section covers:

- **Introducing Exponents and Radicals (Roots) with Variables**
- **Properties of Exponents and Radicals**, **Putting Exponents and Radicals in the Calculator**
- **Rationalizing Radicals**
- **Simplifying Exponential Expressions**
- **Solving Exponential and Radical Equations**
- **Solving Simple Radical Inequalities**
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We briefly talked about exponents in the **Powers, Exponents, Radicals (Roots) and Scientific Notation** section, but we need to go a little bit further in depth and talk about how to do algebra with them. Note that we'll see more radicals in the **Solving Radical Equations and Inequalities** section, and we'll talk about **Factoring with Exponents**, and **Exponential Functions** in the **Exponential Functions** section. Remember that **exponents**, or "raising" a number to a power, are just the number of times that the number (called the **base**) is multiplied by itself. **Radicals** (which comes from the word "root" and means the same thing) means undoing the exponents, or finding out what numbers **multiplied by themselves** comes up with the number. So we remember that $\sqrt{25}=5$, since $5\times 5=25$. Note that we have to remember that when taking the square root (or any even root), we always take the **positive value** (just memorize this).

Introducing Exponents and Radicals (Roots) with Variables

But now that we've learned some algebra, we can do exponential problems with variables in them! So we have $\sqrt{x^2}=x$ (actually $\sqrt{x^2}=|x|$ since ***x*** can be negative) since $x\times x=x^2$. We also learned that taking the square root of a number is the same as raising it to $\frac{1}{2}$, so $x^{\frac{1}{2}}=\sqrt{x}$. Also, remember that when we take the square root, there's an invisible 2 in the radical, like this: $\sqrt[2]{x}$. Also note that what's under the radical sign is called the **radicand** (***x*** in the previous example), and for the **nth root**, the **index** is **n** (**2**, in the previous example, since it's a square root). With a **negative exponent**, there's nothing to do with negative numbers! You move the base from the numerator to the denominator (or denominator to numerator) and make it positive! So if you have a base with a negative number that's not a fraction, put 1 over it and make the exponent positive. And if the negative exponent is on the outside parentheses of a fraction, take the **reciprocal** of the fraction and make the exponent positive. Some examples:

$x^{-2} = \left(\frac{1}{x}\right)^2$ and $\left(\frac{y}{x}\right)^{-4} = \left(\frac{x}{y}\right)^4$. Just a note that we're only dealing with **real numbers** at this point; later we'll learn about **imaginary numbers**, where we can (sort of) take the square root of a negative number.

Properties of Exponents and Radicals

So remember these **basic rules**:

Exponent and Radical Rules	Example	Notes
$x^m = x \cdot x \cdot x \cdot x.....$ (m times)	$2^3 = 2 \cdot 2 \cdot 2 = 8$	***x*** is the base, ***m*** is the exponent
$\sqrt[m]{x} = y$ means $y^m = x$	$\sqrt[3]{8} = 2$, since $2 \cdot 2 \cdot 2 = 2^3 = 8$	***y*** is the base, ***m*** is the root
$\sqrt{x}$ means $\sqrt[2]{x}$	$\sqrt{4} = 2$ means $\sqrt[2]{4} = 2$	When we take a square root, we don't have to put the 2 in the root.
$x^{-m} = \frac{1}{x^m}$ $\frac{1}{x^{-m}} = x^m$ $\left(\frac{x}{y}\right)^{-m} = \left(\frac{y}{x}\right)^m$	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$ $\frac{1}{2^{-2}} = 2^2 = 4$ $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$	Raising a base to a **negative exponent** means taking the **reciprocal** and changing the sign of the exponent.
$a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$ (Doesn't work for imaginary numbers under radicals)	$2\sqrt{3} \times 4\sqrt{5} = 8\sqrt{15}$	When you multiply two radical terms, you can multiply what's on the outside, and also what's in the inside. You can only do this if the **roots are the same** (like square root, cube root).
$\sqrt[\text{even}]{\text{negative number}}$ exists for **imaginary numbers**, but not for **real numbers**.	$\sqrt[4]{-16}$ = no real solution	We can't take the even root of a negative number and get a real number. We can get an "imaginary number", which we'll see later.
$\sqrt[\text{even}]{x^{\text{even}}} = \lvert x \rvert$	$\sqrt{(-2)^2} = \sqrt{4} = 2$	**For an even root, we only take positive value, even if original was negative.** For example, we squared -2 under the square root, but our answer is 2, which is $\lvert -2 \rvert$ (the absolute value of 2).
For $y = x^{\text{even}}$, $y = \pm\sqrt[\text{even}]{x}$	$16 = x^2$; $x = \pm 4$	Since we're taking an **even root**, we have to include both the **positive and negative solutions** in an **equation** with an even exponent. Remember that the square root sign only gives you the positive solutions. This is because **both** the positive root and negative roots work, when raised to that even power,

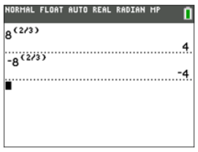
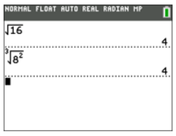
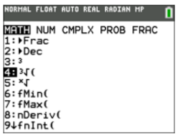
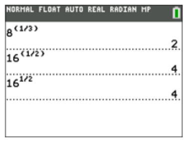
In algebra, we'll need to know these and many other basic rules on how to handle exponents and roots when we work with them. Here are the rules/properties with explanations and examples. In the "proof" column, you'll notice that we're using many of the algebraic properties that we learned in the **Types of Numbers and Algebraic Properties** section, such as the Associate and Commutative properties. Unless otherwise indicated, **assume numbers under radicals with even roots are positive, and numbers in denominators are nonzero**.

Property	Example	"Proof"/Explanation
$(xy)^m = x^m \cdot y^m$	$(xy)^3 = x^3y^3$	$(xy)^3 = xy \cdot xy \cdot xy = (x \cdot x \cdot x) \cdot (y \cdot y \cdot y) = x^3y^3$
$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$	$\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}$	$\left(\frac{x}{y}\right)^4 = \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} = \frac{x \cdot x \cdot x \cdot x}{y \cdot y \cdot y \cdot y} = \frac{x^4}{y^4}$
$x^m \cdot x^n = x^{m+n}$	$x^4 \cdot x^2 = x^6$	$x^4 \cdot x^2 = (x \cdot x \cdot x \cdot x) \cdot (x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$
$\frac{x^m}{x^n} = x^{m-n}$	$\frac{x^5}{x^3} = x^2$	$\frac{x^5}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = \frac{x \cdot x \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = x \cdot x = x^2$
$(x^m)^n = x^{mn}$	$(x^4)^2 = x^8$	$(x^4)^2 = x^4 \cdot x^4 = (x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = x^8$
$x^1 = x$	$473{,}837{,}843^1 = 473{,}837{,}843$	$x^1 = x$ multiplied by itself 1 time $= x$
$x^0 = 1$	$473{,}837{,}843^0 = 1$	$1 = \frac{x^5}{x^5} = x^{5-5} = x^0$
$\frac{1}{x^m} = x^{-m}$	$\frac{1}{3^2} = 3^{-2} = \frac{1}{9}$	$\frac{1}{2^3} = \frac{2^0}{2^3} = 2^{0-3} = 2^{-3}$
$\sqrt[n]{x} = x^{\frac{1}{n}}$	$\sqrt[3]{8} = 8^{\frac{1}{3}} = 2$	Accept that the *n*th root of a base can be written as that base raised to the reciprocal of *n*, or $\frac{1}{n}$.
$\sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y}$	$\sqrt{72} = \sqrt{4 \cdot 9 \cdot 2} = \sqrt{4} \cdot \sqrt{9} \cdot \sqrt{2}$ $= 2 \cdot 3 \cdot \sqrt{2} = 6\sqrt{2}$	$\sqrt{xy} = (xy)^{\frac{1}{2}} = x^{\frac{1}{2}} \cdot y^{\frac{1}{2}} = \sqrt{x} \cdot \sqrt{y}$ (Doesn't work for imaginary numbers under radicals.)
$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$	$\sqrt[3]{\frac{27}{8}} = \frac{\sqrt[3]{27}}{\sqrt[3]{8}} = \frac{3}{2}$	$\sqrt[3]{\frac{x^3}{y^3}} = \sqrt[3]{\frac{x \cdot x \cdot x}{y \cdot y \cdot y}} = \sqrt[3]{\frac{x}{y}} \cdot \sqrt[3]{\frac{x}{y}} \cdot \sqrt[3]{\frac{x}{y}} = \frac{x}{y} = \frac{\sqrt[3]{x^3}}{\sqrt[3]{y^3}}$
$\left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m} = x^{\frac{m}{n}}$ (if *n* is even, $x \geq 0$)	$8^{\frac{2}{3}} = \sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$	Remember that the number inside the root is always on the bottom of the fraction, and the exponent is on top. (I remember that the root is in a cave so it's on the bottom).
n is **odd**: $\left(\sqrt[n]{x}\right)^n = \sqrt[n]{x^n} = x$	$\left(\sqrt[3]{-2}\right)^3 = \sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$	$\left(\sqrt[5]{x}\right)^5 = \sqrt[5]{x^5} = x^{\frac{5}{5}} = x^1 = x$ Note that this works when *n* is even too, if $x \geq 0$.
n is **even**: $\sqrt[n]{x^n} = \lvert x \rvert$	$\sqrt{(-4)^2} = \sqrt{16} = 4$ $\sqrt{(4)^2} = \sqrt{16} = 4$	$\sqrt[4]{(\text{neg number } x)^4} = \sqrt[4]{\text{pos number } x^4}$ $= -(x) = \lvert x \rvert$ Note that for even radicals, if you move any variable to the outside, and it's raised to an odd power, you have to use the absolute value for that variable, if variables aren't assumed to be positive.
$\frac{x}{\sqrt{y}} = \frac{x}{\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{y}} = \frac{x\sqrt{y}}{y}$	$\frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{{}^{2}\cancel{4}\sqrt{2}}{{}^{1}\cancel{2}} = 2\sqrt{2}$	This is called "**rationalizing**" the denominator (getting rid of the radical in the denominator) and is considered better "grammar" in math.

I know this seems like a **lot** to know, but after a lot of practice, they become second nature. You will have to learn the basic properties, but after that, the rest of it will fall in place!

Putting Exponents and Radicals in the Calculator

We can put exponents and radicals in the graphing calculator, using the carrot sign (**^**) to raise a number to something else, the **square root button** to take the square root, or the **MATH** button to get the cube root or *n*th root. Be careful though, because if there's not a perfect square root, the calculator will give you a long decimal number that's not the "**exact value**". The "**exact value**" would be the answer with the root sign in it! Here are some exponent and radical calculator examples:

Keystrokes	Screen	Notes
		To put a radical in the calculator, we can type "**8 ^ (2/3)**". Notice that we need to put parentheses around the 2/3 because the calculator naturally follows the **PEMDAS** order of operations and would otherwise perform the 8 squared first. (This is not necessary in the newer calculators' **MATHPRINT** mode). Notice that when we put a negative on the outside of the 8, it performs the radicals first (cube root of 8, and then squared) and then puts the negative in front of it.
 	 	We can take the square root by using the 2[nd] button, and then the x^2 button. We can also use the **MATH** function to take the cube root or *n*th root. Here we are taking the cube root of 8^2.
		Remember that you can always take the *n*th of any number by raising the number to the $\frac{1}{n}$. Also remember that we don't need the parentheses around the exponent in the newer calculator operating systems (but it won't hurt to have them).

Rationalizing Radicals

Before we work example, let's talk about **rationalizing radical fractions**. In math, sometimes we have to worry about "proper grammar". When radicals, it's improper grammar to have a root on the bottom in a fraction – in the denominator. To fix this, we multiply by a fraction with the bottom radical(s) on both the top and bottom (so the fraction equals 1); this way the bottom radical disappears. Neat trick! Here are some examples:

Math	Notes
$\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{1\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}=\frac{\sqrt{2}}{2}$	Since the $\sqrt{2}$ is on the bottom, we need to get rid of it by multiplying by 1, or $\frac{\sqrt{2}}{\sqrt{2}}$.
$\frac{4}{2\sqrt{3}}=\frac{4}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{4\sqrt{3}}{2\sqrt{3}\cdot\sqrt{3}}=\frac{{}^{2}\not{4}\sqrt{3}}{{}_{1}\not{2}\cdot 3}=\frac{2\sqrt{3}}{3}$	Since the $\sqrt{3}$ is on the bottom, we need to multiply by 1, or $\frac{\sqrt{3}}{\sqrt{3}}$. Note that we didn't need to multiply by $2\sqrt{3}$, only by the radical.
$\frac{5}{2\sqrt[4]{3}}=\frac{5}{2\sqrt[4]{3}}\cdot\frac{(\sqrt[4]{3})^3}{(\sqrt[4]{3})^3}=\frac{5(\sqrt[4]{3})^3}{2(\sqrt[4]{3})^1(\sqrt[4]{3})^3}=\frac{5(\sqrt[4]{3})^3}{2(\sqrt[4]{3})^4}=\frac{5(\sqrt[4]{3})^3}{2\cdot 3}=\frac{5(\sqrt[4]{3})^3}{6}$	Since we have the 4[th] root of 3 on the bottom ($\sqrt[4]{3}$), we need to multiply by that number **cubed**, to eliminate the 4[th] root. Don't worry if you don't totally get this now!

Simplifying Exponential Expressions

There are five main things you'll have to do to simplify exponents and radicals. For the purpose of the examples below, we are assuming that **variables in radicals are non-negative, and denominators are nonzero**.

- **get rid of parentheses ()**. Remember that when an exponential expression is raised to another exponent, you multiply exponents. Also remember when you are multiplying numbers and variables and the whole thing is raised to an exponent, you can remove parentheses and "push through" the exponent. Example: $(2x^3y)^2=4x^6y^2$
- **combine bases to combine exponents**. You should add exponents of common bases if you multiplying, and subtract exponents of common bases if you are dividing (you can subtract "up", or subtract "down" to get the positive exponent as you'll see). Sometimes you have to match the bases first in order to combine exponents – see last example below. Examples: $a^2a^3=a^5$ $\quad \frac{a^5}{a^3}=a^2$ $\quad \frac{a^3}{a^5}=\frac{1}{a^2}$
- **get rid of negative exponents**. To get rid of negative exponents, you can simply move a negative exponent in the denominator to the numerator and make it positive, or vice versa. Examples: $a^{-4}=\frac{1}{a^4}$ $\quad \left(\frac{a}{b}\right)^{-2}=\left(\frac{b}{a}\right)^2$
- **simplify any numbers** (like $\sqrt{4}$ = 2). Also, remember to simplify radicals by taking out any factors of perfect squares (under a square root), cubes (under a cube root), and so on. Example: $\sqrt{50x^2}=\sqrt{25\cdot 2\cdot x^2}=\sqrt{25}\cdot\sqrt{2}\cdot\sqrt{x^2}=5x\sqrt{2}$
- **combine any like terms**. If you're adding or subtracting terms with the same numbers (coefficients) and/or variables, you can put these together. Almost think of a **radical expression** (like $\sqrt{2}$) **like another variable**. Example: $4x^2\sqrt{2}-2x^2\sqrt{2}=2x^2\sqrt{2}$. **Remember that, for the variables, we can divide the exponents inside by the root index – if it goes in exactly, we can take the variable to the outside; if there are any remainders, we have to leave the variables under the root sign.** For example, $\sqrt[3]{x^5y^{12}}=x^1y^4\sqrt[3]{x^2}=xy^4\sqrt[3]{x^2}$, since 5 divided by 3 is 1, with 2 left over (for the ***x***), and 12 divided by 3 is 4 (for the ***y***).

Now let's put it altogether. Here are some (difficult) examples. Just remember that you have to be really, really careful doing these!

Expression	Simplification	Explanation
$\left(9x^3y\right)^2$	$\left(9x^3y\right)^2=9^2x^6y^2=81x^6y^2$	"Push through" the exponent when eliminating the parentheses.
$\left(6a^{-2}b\right)\left(\frac{2ab^3}{4a^3}\right)^2$	$\left(6a^{-2}b\right)\left(\frac{2ab^3}{4a^3}\right)^2$ $=6a^{-2}b\cdot\frac{4a^2b^6}{16a^6}=\frac{24a^0b^7}{16a^6}=\frac{3b^7}{2a^6}$	Eliminate the parentheses with the squared first. Then combine variables and add or subtract exponents. Remember that a^0 is 1.
$\frac{34n^{2x+y}}{17n^{x-y}}$	$\frac{34n^{2x+y}}{17n^{x-y}}=2n^{(2x+y)-(x-y)}$ $=2n^{2x-x+y-(-y)}=2n^{x+2y}$	Divide the actual numbers (coefficients) to get 2; subtract exponents with base of *n*.
$\frac{\left(4a^{-3}b^2\right)^{-2}\left(a^3b^{-1}\right)^3}{\left(-2a^{-3}\right)^2}$	$\frac{\left(4a^{-3}b^2\right)^{-2}\left(a^3b^{-1}\right)^3}{\left(-2a^{-3}\right)^2}=\frac{\left(a^3b^{-1}\right)^3}{\left(4a^{-3}b^2\right)^2\left(-2a^{-3}\right)^2}$ $=\frac{a^9b^{-3}}{\left(16a^{-6}b^4\right)\left(4a^{-6}\right)}=\frac{a^9b^{-3}}{64a^{-12}b^4}$ $=\frac{a^{9-(-12)}}{64b^{4-(-3)}}=\frac{a^{21}}{64b^7}$	Move what's inside the negative exponent down first and make exponent positive. Then get rid of parentheses first, by pushing the exponents through. Notice that, since we wanted to end up with positive exponents, we kept the positive exponents where they were in the fraction, and subtracted up or down to turn the negative exponents positive.
$\left(-8\right)^{\frac{2}{3}}$	$\left(-8\right)^{\frac{2}{3}}=\left(\sqrt[3]{-8}\right)^2=\left(-2\right)^2=4$	Remember that the bottom of the fraction is what goes in the root, and we typically take the root first. We could also put this in our calculator!
$\left(\frac{a^9}{27}\right)^{-\frac{2}{3}}$	$\left(\frac{a^9}{27}\right)^{-\frac{2}{3}}=\left(\frac{27}{a^9}\right)^{\frac{2}{3}}=\frac{27^{\frac{2}{3}}}{\left(a^9\right)^{\frac{2}{3}}}=\frac{\left(\sqrt[3]{27}\right)^2}{a^{\frac{18}{3}}}$ $=\frac{\left(\sqrt[3]{27}\right)^2}{a^6}=\frac{3^2}{a^6}=\frac{9}{a^6}$	Flip fraction first to get rid of negative exponent. "Carry through" the exponent to both the top and bottom of the fraction and remember that the cube root of 27 is 3.
$\left(\frac{2^{-1}+2^{-2}}{2^{-4}}\right)^{-1}$	$\left(\frac{2^{-1}+2^{-2}}{2^{-4}}\right)^{-1}=\frac{2^{-4}}{2^{-1}+2^{-2}}=\frac{\frac{1}{2^4}}{\frac{1}{2}+\frac{1}{4}}$ $=\frac{\frac{1}{2^4}}{\frac{2}{4}+\frac{1}{4}}=\frac{\frac{1}{2^4}}{\frac{3}{4}}=\frac{1}{{}_4\cancel{16}}\cdot\frac{\cancel{4}^1}{3}=\frac{1}{12}$	We can't push the -1 exponent through since we're **adding** on the top and not multiplying. So flip the fraction, and then do the math with each term separately.
$\left(\frac{2^{-1}+2^{-2}}{2^{-4}}\right)^{-1}$	$\left(\frac{2^{-1}+2^{-2}}{2^{-4}}\right)^{-1}=\frac{2^{-4}}{2^{-1}+2^{-2}}\times\frac{2^4}{2^4}$ $=\frac{\left(2^{-4}\right)\left(2^4\right)}{2^{-1}\left(2^4\right)+2^{-2}\left(2^4\right)}=\frac{1}{2^3+2^2}=\frac{1}{12}$	**Easier way to do problems like this:** If the same base is on the top and bottom, multiply by 1, with the numerator and denominator being that base with the positive exponent that is the opposite of the smallest (largest negative) of all the exponents; in our case, multiply by $\frac{2^4}{2^4}$, since -4 is the smallest exponent.

Here are even more examples. **Assume variables under radicals are non-negative**.

Expression	Simplification	Explanation
$\sqrt[4]{64a^7b^8}$	$\sqrt[4]{64a^7b^8}=\left(\sqrt[4]{64}\right)\sqrt[4]{a^7b^8}$ $=\left(\sqrt[4]{16}\right)\left(\sqrt[4]{4}\right)\left(\sqrt[4]{a^7}\right)\sqrt[4]{b^8}$ $=2\left(\sqrt[4]{4}\right)a^1\sqrt[4]{a^3}b^2$ $=2ab^2\sqrt[4]{4a^3}$	**METHOD 1:** Separate the numbers and variables. With $\sqrt[4]{64}$, we factor 64 into 16 and 4, since $\sqrt[4]{16}=2$. The 4th root of a^7 is $a\sqrt[4]{a^3}$, since 4 goes into 7 **one** time (so we can take one a out), and there's 3 **left over** (to get the a^3). The 4th root of b^8 is b^2, since 4 goes into 8 exactly 2 times. Then we can put it all together, combining the radical.
$\sqrt[4]{64a^7b^8}$	$\sqrt[4]{64a^7b^8}=\left(64a^7b^8\right)^{\frac{1}{4}}$ $=(64)^{\frac{1}{4}}\left(a^7\right)^{\frac{1}{4}}\left(b^8\right)^{\frac{1}{4}}$ $=(16)^{\frac{1}{4}}(4)^{\frac{1}{4}}a^{\frac{7}{4}}b^2$ $=2(4)^{\frac{1}{4}}a^{\frac{4}{4}}a^{\frac{3}{4}}b^2$ $=2ab^2\sqrt[4]{4a^3}$	**METHOD 2:** Turn the fourth root into an exponential fraction and "carry it through". Then raise each number and variable to the $\frac{1}{4}$ and remember if the fraction is over 1 (in the case of both a and b), we have to take at least some of it to the outside (all of it to the outside, in the case of the b). With $64^{\frac{1}{4}}$, we factor it into 16 and 4, since $16^{\frac{1}{4}}$ is 2.
$6x^2\sqrt{48y^2}-4y\sqrt{27x^4}$	$6x^2\sqrt{48y^2}-4y\sqrt{27x^4}$ $=6x^2y\sqrt{16\cdot 3}-4x^2y\sqrt{9\cdot 3}$ $=6\cdot 4\cdot x^2y\sqrt{3}-3\cdot 4x^2y\sqrt{3}$ $=24\sqrt{3}x^2y-12\sqrt{3}x^2y$ $=12\sqrt{3}x^2y$	Simplify the roots (both numbers and variables) by taking out squares. Then, combine like terms, where you need to have the same root and variables. We could have turned the roots into fractional exponents and gotten the same answer – it's a matter of preference.
$\sqrt[4]{\frac{x^6y^4}{162z^5}}$	$\sqrt[4]{\frac{x^6y^4}{162z^5}}=\frac{\sqrt[4]{x^6y^4}}{\sqrt[4]{(81)(2)z^5}}=\frac{xy\sqrt[4]{x^2}}{3z\sqrt[4]{2z}}$ $=\frac{xy\sqrt[4]{x^2}}{3z\sqrt[4]{2z}}\cdot\frac{\sqrt[4]{(2z)^3}}{\sqrt[4]{(2z)^3}}=\frac{xy\sqrt[4]{x^2}\sqrt[4]{8z^3}}{3z\sqrt[4]{(2z)^4}}$ $=\frac{xy\sqrt[4]{8x^2z^3}}{3z(2z)}=\frac{xy\sqrt[4]{8x^2z^3}}{6z^2}$	This one's pretty complicated since we have to **simplify** and **rationalize**. I like to separate the numerator and the denominator first, and then take out everything that can be raised to the 4th under the radicals. (Remember the trick with variables – like with the x's on the top, 4 goes into 6, 1 time, with 2 left over, so x is on the outside and x^2 is on the inside.) Then, to rationalize, since we have a 4th root, we have to multiply by a radical that has the 3rd root on top and bottom.
$32^{\frac{3}{5}}\cdot 81^{\frac{1}{4}}\cdot 27^{-\frac{1}{3}}$	$32^{\frac{3}{5}}\cdot 81^{\frac{1}{4}}\cdot 27^{-\frac{1}{3}}$ $=(2)^3\cdot(3)^1\cdot(3)^{-1}$ $=8\cdot 3\cdot\frac{1}{3}=8$	Each root had a "perfect" answer, so we took the roots first. Then we applied the exponents, and then just multiplied across. We could put this one in the calculator (using parentheses around the fractional roots) too.
$9^{x-2}\cdot 3^{x-1}$	$9^{x-2}\cdot 3^{x-1}=\left(3^2\right)^{x-2}\cdot 3^{x-1}$ $=3^{2(x-2)}\cdot 3^{x-1}=3^{2x-4}\cdot 3^{x-1}$ $=3^{2x-4+x-1}=3^{3x-5}$	We will visit this concept again later, but to add the exponents, we can get a common **base**, which is 3.

If we **don't assume variables under the radicals are non-negative**, we have to be careful with the signs and include **absolute values for even radicals**. Here's an example:

Expression	Simplification	Explanation
$\sqrt{45a^3b^2}$	$\sqrt{45a^3b^2}=\left(\sqrt{45}\right)\sqrt{a^3b^2}$ $=\left(\sqrt{9}\right)\left(\sqrt{5}\right)\left(\sqrt{a^3}\right)\sqrt{b^2}$ $=3\left(\sqrt{5}\right)\left(\sqrt{a^2}\right)\left(\sqrt{a}\right)\sqrt{b^2}$ $=3\left(\sqrt{5}\right)\lvert a\rvert\cdot\sqrt{a}\cdot\lvert b\rvert$ $=3\lvert a\rvert\lvert b\rvert\left(\sqrt{5a}\right)$	Separate the numbers and variables. With $\sqrt{45}$, we factor 45 into 9 and 5, since $\sqrt{9}=3$. The square root of a^3 is $a\sqrt{a}$, since 2 goes into 3 **one** time (so we can take one a out), and there's 1 **left over** (to get the a). But, if we can have a **negative** a, when we square it and then take the square root, it turns into a positive. So in this case, a^3 is $\lvert a\rvert\sqrt{a}$. Similarly, $\sqrt{b^2}$ is $\lvert b\rvert$. Then we can put it all together, combining the radical.

For all these examples, see how we're doing the same steps over and over again – just with different problems? If you don't get them at first, don't worry; just try to go over them again. You'll get it! And don't forget that there are many ways to arrive at the same answers!

Solving Exponential and Radical Equations

(We'll see more of these types of problems **here** in the **Solving Radical Equations and Inequalities** section). Now that we know about exponents and roots with variables, we can solve equations that involve them. **The trick is to get rid of the exponents, we need to take radicals of both sides, and to get rid of radicals, we need to raise both sides of the equation to that power.** You have to be a little careful, especially with **even exponents and roots** (the "**evil evens**"), and also when the even exponents are on the **top** of a fractional exponent (this will become the root part when we solve). When we solve for variables with **even exponents**, we most likely will get **multiple solutions**, since when we square positive or negative numbers, we get positive numbers. Also, all the answers we get may not work, since we **can't take the even roots of negative numbers**. **So it's a good idea to always check our answers when we solve for roots (especially even roots)!** Let's first try some equations with **odd exponents and roots**, since these are a little more straightforward. (Notice when we have **fractional** exponents, the radical is still **odd** when the **numerator** is odd).

Math	Notes
$x^3-1=26$ $x^3=27$ $\sqrt[3]{x^3}=\sqrt[3]{27}$ $x=3$	Move all the constants (numbers) to the right. To get rid of the x^3, you can take the cube root of each side. (Note that we could have also raised each side to the $\frac{1}{3}$power.) Remember that when we cube a cube root, we end up with what's under the root sign. Since the root is odd, we don't have to worry about the signs. Let's check our answer: $3^3-1=27-1=26$ √
$2\sqrt[3]{x+2}=6$ $\sqrt[3]{x+2}=3$ $\left(\sqrt[3]{x+2}\right)^3=3^3$ $x+2=27$ $x=25$	First, divide both sides by 2; always try to simplify before solving with radicals. To get rid of the radical on the left hand side, we can cube both sides. Then we can solve for ***x***. Let's check our answer: $2\sqrt[3]{25+2}=2(3)=6$ √
$(y+2)^{\frac{3}{2}}=8$ $\left((y+2)^{\frac{3}{2}}\right)^{\frac{2}{3}}=8^{\frac{2}{3}}$ $(y+2)^{\frac{3}{2}\times\frac{2}{3}}=8^{\frac{2}{3}}$ $y+2=\left(\sqrt[3]{8}\right)^2=2^2=4$ $y+2=4$ $y=2$	We can raise both sides to the same number. So we want to raise both sides to the **reciprocal of the exponent on the left**, so it turns into a 1. Remember that when you raise an exponent to another exponent, you can multiply the two exponents. This is a neat trick! To raise 8 to the $\frac{2}{3}$, we can either do this in a calculator, or take the cube root of 8 and square it. Then we can solve for ***y*** by subtracting 2 from each side. Let's check our answer: $(2+2)^{\frac{3}{2}}=(4)^{\frac{3}{2}}=\left(\sqrt{4}\right)^3=2^3=8$ √ (Notice in this case, that we have to make sure $(y+2)$ is positive since we are taking an even root, but when we work the problem, we can be assured it is, since we are squaring the right hand side. If the problem were $(y+2)^{\frac{3}{2}}=-8$, for example, we would have no solution.)
$4\sqrt[3]{x}=2\sqrt[3]{x+7}$ $2\sqrt[3]{x}=\sqrt[3]{x+7}$ $\left(2\sqrt[3]{x}\right)^3=\left(\sqrt[3]{x+7}\right)^3$ $8x=x+7$ $7x=7$ $x=1$	Since we have the cube root on each side, we can simply cube each side. It's always easier to simply (for example, **divide both sides by 2**) first, but you don't have to. Be careful to make sure you cube all the numbers (and anything else on that side) too. Then we can solve for ***x***. Let's check our answer: $4\sqrt[3]{1}=2\sqrt[3]{1+7}$? $4=4$ √

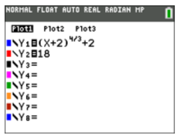
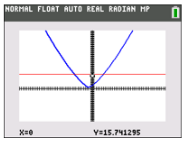
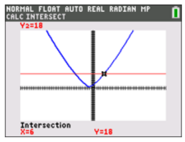
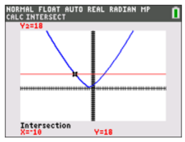
Now let's solve equations with **even roots**. Note that when we take the even root (like the square root) of both sides, we have to include the **positive** and the **negative** solutions of the roots. (Notice when we have **fractional** exponents, the radical is still **even** when the **numerator** is even.) Again, when the original problem contains an even root sign, we need to check our answers to make sure we have end up with **no negative numbers under the even root sign** (no negative radicands). Also, if we **have squared both sides** (or raised both sides to an even exponent), we need to **check our answers to see if they work**. The solutions that don't work when you put them back in the original equation are called **extraneous solutions**. Again, we'll see more of these types of problems in the **Solving Radical Equations and Inequalities** section.

Math	Notes
$x^2+9=5$ $x^2=-4$ $\varnothing$ or no solution	Since we can never square any real number and end up with a negative number, there is no real solution for this equation.
$x^2-1=24$ $x^2=25$ $x=\pm 5$	Since we're taking an even root, we have to include both the **positive and negative solutions**. Remember that the square root sign only gives you the positive solutions. We need to check our answers: $(5)^2-1=24$ √ $(-5)^2-1=24$ √
$\sqrt[4]{x+3}=2$ $\left(\sqrt[4]{x+3}\right)^4=2^4$ $x+3=16$ $x=13$	We can "undo" the fourth root by raising both sides to the forth. We don't need to worry about plus and minuses since we're not taking the root of a number. We need to check our answer to make sure there are **no negative numbers under the even radical** and also still check the answers **since we raised both sides to the 4th power**: $\sqrt[4]{13+3}=\sqrt[4]{16}=2$ √
$4\sqrt{x-1}=\sqrt{x+1}$ $\left(4\sqrt{x-1}\right)^2=\left(\sqrt{x+1}\right)^2$ $4^2(x-1)=(x+1)$ $16x-16=x+1$ $15x=17; \quad x=\frac{17}{15}$	Since we have square roots on both sides, we can simply square both sides to get rid of them. We have to make sure we square the 4 too. Then we solve for ***x*** to get $\frac{17}{15}$. We have to make sure our answers don't produce any negative numbers under the square root; this looks good. Also, since we squared both sides, let's check our answer: $4\sqrt{\frac{2}{15}}=\sqrt{\frac{32}{15}}$? $4\sqrt{\frac{2}{15}}=\sqrt{(16)(2)\frac{1}{15}}$? $4\sqrt{\frac{2}{15}}=4\sqrt{\frac{2}{15}}$ √
$(x+2)^{\frac{4}{3}}+2=18$ $\left((x+2)^{\frac{4}{3}}\right)^{\frac{3}{4}}=16^{\frac{3}{4}}$ $x+2=\pm 2^3$ $x=\pm 2^3-2$ $x=8-2=6$ and $x=-8-2=-10$	In the case of a **fractional exponent** on the left (near the variable), if the even number is **on the top** of the fraction, you have to take the **positive and negative** solutions. We also have to make sure the number on the right that we're raising to an exponent is **positive**, or there would be no answers. Let's check our answers: $(6+2)^{\frac{4}{3}}+2=\left(\sqrt[3]{8}\right)^4+2=2^4+2=18$ √ $(-10+2)^{\frac{4}{3}}+2=\left(\sqrt[3]{-8}\right)^4+2=(-2)^4+2=18$ √
$\sqrt{2-x}=\sqrt{x-4}$ $\left(\sqrt{2-x}\right)^2=\left(\sqrt{x-4}\right)^2$ $2-x=x-4$ $2x=6$ $x=3$ Doesn't work, so no solution!	We correctly solved the equation, but notice that when we plug in 3 in the first radical (and the second one, too!), we get a **negative number** $(2-3=-1)$. So we have to "throw away" our answer and the correct answer is "**no solution**" or $\varnothing$. **This shows us that we must plug in our answer when we're dealing with even roots!**

And here's one more where we're solving for one variable in terms of the other variables:

Math	Notes
Solve for y_2 : $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ $d^2=(x_1-x_2)^2+(y_1-y_2)^2$ $d^2-(x_1-x_2)^2=(y_1-y_2)^2$ $\pm\sqrt{d^2-(x_1-x_2)^2}=y_1-y_2$ $y_2=y_1\pm\sqrt{d^2-(x_1-x_2)^2}$	We keep moving variables around until we have y_2 on one side. Since we have to get y_2 by itself, we first have to take the square root of each side (and don't forget to take the plus and the minus). Then we solve for y_2. Notice that when we moved the $\pm$ to the other side, it's still a $\pm$.

Note: You can also check your answers using a **graphing calculator** by putting in what's on the left of the = sign in "Y_1 =" and what's to the right of the equal sign in "Y_2 =". You can then use the intersection feature to find the solution(s); the solution(s) will be what ***x*** is at that point. Here are those instructions again, using an example from above:

Instructions	Screen
Push **Y =** and enter the two equations in $\mathbf{Y_1}$ **=** and $\mathbf{Y_2}$ **=**, respectively. Push **GRAPH**. You may need to hit "**ZOOM 6**" (ZStandard) and/or "**ZOOM 0**" (ZoomFit) to make sure you see the lines crossing in the graph. (You can also use the **WINDOW** button to change the minimum and maximum values of your x and y values.) I also used "**ZOOM 3**" (Zoom Out) **ENTER** to see the intersections a little better. (You may have to do this a few times). You can see that we have **two points of intersections**; therefore, we have **two solutions**.	 
To get the first point of intersection, push "2[nd] **TRACE**" (**CALC**), and then either push **5**, or move cursor down to **intersect**. You should see "**First curve?**" at the bottom. Then push **ENTER, ENTER, ENTER**. You'll see the first point of intersection that it found is where ***x* = 6**. To find the other point of intersection, we need to move the cursor closer to that point, so press "**TRACE**" and move the cursor closer to the other point of intersection (it should follow along one of the curves). Then do the step above again with "2[nd] **TRACE**" (**CALC**), **5**, **ENTER, ENTER, ENTER**. You should see the second solution at ***x* = – 10**. These are the answers we got above!	 

Solving Simple Radical Inequalities

Note again that we'll see more problems like these, including how to use sign charts with solving radical inequalities here in the **Solving Radical Equations and Inequalities** section. Just like we had to solve linear inequalities, we also have to learn how to solve inequalities that involve exponents and radicals (roots). We'll do this pretty much the same way, but again, we need to be careful with **multiplying and dividing by anything negative**, where we have to change **the direction of the inequality sign**. We also have to be careful that our answer still keeps what's **under an even radical** to be **positive**, so we have to create another inequality and set what's under the even radical to greater than or equal to 0, solve for ***x***, and take the intersection of both solutions. The reason we take the intersection of the two solutions is because **both** must work. With **odd roots**, we don't have to worry – we just raise each side that power, and solve! Here are some examples:

Math	Notes
$\sqrt{x+2}\le 4$ $\sqrt{x+2}\le 4$ and $x+2\ge 0$ $\left(\sqrt{x+2}\right)^2\le 4^2$ and $x+2\ge 0$ $x+2\le 16$ and $x\ge 0-2$ $x\le 14$ and $x\ge -2$ $\{x:-2\le x\le 14\}$ or $[-2,14]$	We actually have to solve two inequalities, since our ***x*** must work in the original, but also work so **anything under the even radical** is **positive**. So we have to solve two inequalities: one with the original problem and one to make sure the **radical in the original problem is ≥ 0.** To get rid of the square roots, we square each side, and we can leave the inequality signs the same since we're **multiplying by positive numbers**. Then we just solve for ***x***, just like we would for an equation. We need to check our answer by trying random numbers in our solution (like ***x* = 2**) in the **original inequality** (which works). We also need to try numbers outside our solution (like ***x* = –6** and ***x* = 20**) and see that they don't work.
$\sqrt{5x-16}<\sqrt{2x-4}$ $\left(\sqrt{5x-16}\right)^2<\left(\sqrt{2x-4}\right)^2$ $5x-16<2x-4$ $3x<12$ $x<4$ also: $5x-16\ge 0$ and $2x-4\ge 0$ $x\ge\frac{16}{5}$ and $x\ge 2$ $x<4\cap x\ge\frac{16}{5}\cap x\ge 2$ $\{x:\frac{16}{5}\le x<4\}$ or $\left[\frac{16}{5},4\right)$	Now we have to solve **three inequalities:** one for the main problem, and one each for the even root radicands, that have to be ≥ 0. We need to take the **intersection** (all must work) of the inequalities: $x<4$ and $x\ge\frac{16}{5}$ and $x\ge 2$. This will give us $\frac{16}{5}\le x<4$. (Try it yourself on a number line). Watch out for the hard and soft brackets. We need to check our answer by trying ***x* = 3.5** in the **original inequality** (which works) and ***x* = 3** or ***x* = 5** (which don't work).
$\sqrt{x+6}\le -2$ $\{\}$ or $\varnothing$	Before we even need to get started with this inequality, we can notice that the **square root of anything** can never be less than ≤ -2 (**or < 0**), by definition. So we know right away that the answer is no solution, or {}, or Ø.
$\sqrt[3]{x-3}>4$ $\left(\sqrt[3]{x-3}\right)^3>4^3$ $x-3>64$ $x>67$	With odd roots, we don't have to worry about checking underneath the radical sign, since we could have positive or negative numbers as a radicand.

Learn these rules, and practice, practice, practice!