

Variation Mode and Effect Analysis: an application to fatigue life prediction

Pär Johannesson^{1,*}, Thomas Svensson², Leif Samuelsson³,
Bo Bergman⁴, and Jacques de Maré^{5,6,1}

¹*Fraunhofer-Chalmers Research Centre for Industrial Mathematics, Sweden*

²*SP Technical Research Institute of Sweden*

³*Volvo Aero Corporation, Sweden*

⁴*Quality Science, Chalmers, Sweden*

⁵*Mathematical Statistics, Chalmers, Sweden*

⁶*Mathematical Statistics, Gothenburg University, Sweden*

Abstract

We will present an application of the probabilistic branch of *Variation Mode and Effect Analysis* (VMEA) implemented as a *first order, second moment reliability method*. First order means that the failure function is approximated to be linear with respect to the main influencing variables, while second moment means that only means and variances are taken into account in the statistical procedure.

We study the fatigue life of an air engine component and aim at a safety margin that takes all sources of scatter and uncertainty into account. *Scatter* is defined as random variation due to natural causes, such as non-homogeneous material, geometry variation within tolerances, load variation in usage, and other uncontrolled variation. *Uncertainty* is defined as unknown systematic errors, such as model errors in the numerical calculation of fatigue life, statistical errors in estimates of parameters, and unknown usage profile.

By defining also uncertainties as random variables, the whole safety margin problem is put into a common framework of second order statistics, with the Gauss' approximation formula as the main tool. By using a simple log transformation, the failure function is regarded as linear enough for a proper estimate of the scatter and uncertainty contributions based on the first order approximation.

Thus, the final estimated variance of the logarithmic life is obtained through summing the variance contributions of all sources of scatter and uncertainty, and it represents the total uncertainty in the life prediction. Motivated by the central limit theorem this logarithmic life random variable may be regarded as normally distributed, which gives possibilities to calculate relevant safety margins that, in turn, is transformed back into fatigue life margins for comparisons with demands.

Keywords: Probabilistic VMEA, fatigue life, life prediction, safety factor.

*Corresponding author: Pär Johannesson, Fraunhofer-Chalmers Research Centre for Industrial Mathematics; Chalmers Science Park, SE-412 88 Göteborg, Sweden
E-mail: Par.Johannesson@fcc.chalmers.se

1. Introduction

An important goal of engineering design is to get a reliable system, structure or component. One such well-established method is FMEA (Failure Mode and Effect Analysis), where the aim is to identify possible failure modes and evaluate their effect. A general design philosophy, within robust design, is to make designs that avoid failure modes as much as possible, see e.g. [Davis, 2006]. Further, it is important that the design is robust against different sources of unavoidable variation. A general methodology called VMEA (Variation Mode and Effect Analysis) has been developed in order to deal with this problem, see [Johansson, et al., 2006] and [Chakhunashvili, et al., 2006]. The VMEA is split into three different levels; 1) basic VMEA, in the early design stage, when we only have vague knowledge about the variation, and the goal is to compare different design concepts, 2) advanced VMEA, further in the design process when we can better judge the sources of variation, and 3) probabilistic VMEA, in the later design stages where we have more detailed information about the structure and the sources of variation, and the goal is to be able to assess the reliability.

This paper treats the third level, the probabilistic VMEA, and we suggest a simple model, also used in [Svensson, 1997], for assessing the total uncertainty in a fatigue life prediction, where we consider different sources of variation, as well as statistical uncertainties and model uncertainties. The model may be written as

$$Y = \hat{Y} + X_1 + X_2 + \dots + X_p + Z_1 + Z_2 + \dots + Z_q \quad (1)$$

where the X_k 's and the Z_k 's are random variables representing different sources of scatter or uncertainty, respectively. In our case it is appropriate to study the logarithmic life, thus $Y = \ln N$, where N is the life. The prediction of the logarithmic life $\hat{Y} = \ln \hat{N}$ may be a complicated function, e.g. defined through finite element software; however the analysis of the prediction uncertainty is based on a linearization of the function, making use of only the sensitivity coefficients. Further, for reliability assessments the log-life, $Y = \ln N$, is approximated by a normal distribution. The methodology will be discussed using a case study of a low pressure shaft in a jet engine, see Figure 1.



Figure 1. Low pressure shaft in a jet engine.

The classification of sources of prediction uncertainty is discussed in Section 2. In Section 3 the problem of life prediction is studied where a linear model of the logarithmic life is introduced and includes the different kinds of prediction uncertainties. An application of the method of estimating the size of different prediction

uncertainties is given in Section 4. In the following section the quantification of the sum of the uncertainties is used to construct a prediction interval which is a rational way to construct safety factors. If further experiments are carried out a possibility to update the prediction interval is proposed in Section 6. In particular the experiments are used for eliminating a possible systematic prediction error. The last section contains concluding remarks about the applicability of the method and its similarity to the first order second moment method.

2. Scatter & uncertainty

There are various ways in which the types of variation might be classified, see e.g. [Melchers, 1999], [Ditlevsen & Madsen, 2005] and [Lodeby, 2000]. The first way is to distinguish between *aleatory* uncertainties and *epistemic* uncertainties. The first one refers to the underlying, intrinsic uncertainties, e.g. the scatter in fatigue life and the variation within a class of customers. The latter one refers to the uncertainties which can be reduced by means of additional data or information, better modelling and better parameter estimation methods.

Here we will use the terminology *scatter* as being the aleatory uncertainties, and just *uncertainty* as being the epistemic uncertainties. In our approach, we will focus on the three kinds of uncertainties mentioned by [Ditlevsen & Madsen, 2005], and denote them by

- ***Scatter*** or physical uncertainty is that identified with the inherent random nature of the phenomenon, e.g. the variation in strength between different components.
- ***Statistical uncertainty*** is that associated with the uncertainty due to statistical estimators of physical model parameters based on available data, e.g. estimation of parameters in the Coffin-Manson model for life based on fatigue tests.
- ***Model uncertainty*** is that associated with the use of one (or more) simplified relationship to represent the 'real' relationship or phenomenon of interest, e.g. a finite element model used for calculating stresses, is only a model for the 'real' stress state.

Another important kind of uncertainties is what can be called call uncertainty due to human factors. These are not treated here, but must be controlled by other means, which is discussed in e.g. [Melchers, 1999]. In our analysis we assume that the phenomenon is predictable. This is the case if the system and the distribution of the random variables don't change with time.

3. A simple approach to probabilistic VMEA

Here we will present a simple model for the prediction uncertainty based on a summation of contributions from different sources on scatter and uncertainty. We will discuss it in terms of fatigue life prediction, but it can easily be adapted to other situations, e.g. prediction of maximum stress or maximum defect sizes.

3.1. Model for uncertainty in life predictions

We will study the prediction error of the logarithmic life prediction

$$\hat{Y} = \ln \hat{N} = f(\Psi, \hat{\theta}, \hat{\mathbf{X}}) \quad (2)$$

where $f(\Psi, \hat{\theta}, \hat{\mathbf{X}})$ is our model for the life which involves the damage driving parameter, Ψ (e.g. stress, strain or force), the estimated parameter vector, $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_r)$, and the modelled scatter, $\hat{\mathbf{X}}$. The prediction error can be written as

$$\hat{e} = Y - \hat{Y} = \ln N - \ln \hat{N} = g(\Psi, \tilde{\mathbf{X}}) - f(\Psi, \hat{\theta}, \hat{\mathbf{X}}) \quad (3)$$

where $g(\Psi, \tilde{\mathbf{X}})$ is the actual relation for the log-life depending on the damage driving parameter, Ψ , and the involved scatter, $\tilde{\mathbf{X}}$. The next step is to approximate the prediction error by a sum

$$\hat{e} = X_1 + X_2 + \dots + X_p + Z_1 + Z_2 + \dots + Z_q \quad (4)$$

where the quantities $\mathbf{X} = (X_1, \dots, X_p)$ and $\mathbf{Z} = (Z_1, \dots, Z_q)$ represent different types of scatters and uncertainties, respectively. The random quantities X_i and Z_j are assumed to have zero mean and variances τ_i^2 and δ_j^2 , respectively. In the analysis we only use the variances and covariances of the X_i 's and Z_j 's, but not their exact distributions, which are often not known to the designer. In some situations it is more natural to estimate the scatter or uncertainty in some quantity that is related to log-life, and then use a sensitivity coefficient to get its effect on the log-life. One such example is the scatter in life due to geometry variations due to tolerances, where it is easier to estimate the scatter in stress, say τ'_i , which is then transferred via a sensitivity coefficient to the scatter in life, $\tau_i = |c_i| \cdot \tau'_i$. This is motivated by Gauss approximation formula for the transfer function $\ln N = h(S)$ from stress to log-life, which gives

$$\ln N = h(S) \approx h(s_0) + c_i(S - s_0) \quad \text{with} \quad c_i = \left. \frac{dh}{dS} \right|_{S=s_0} \quad (5)$$

where s_0 is the stress corresponding to the nominal values. Hence, the variance for the log-life is approximated by $\tau_i^2 = c_i^2 \cdot \tau_i'^2$.

When constructing reliability measures in Section 5, we will make use of a normal distribution approximation on the log-life. Theoretically this is motivated through the central limit theorem (CLT), which in its simplest form states that the sum

$S_n = (X_1 + X_2 + \dots + X_n)/\sqrt{n}$ converges, as n tends to infinity, to a normal distribution with zero mean and variance σ^2 , if the X_k 's are independent and equally distributed with mean zero and finite variance σ^2 . Note that the convergence is

regardless of the parent distribution of X_k . The theorem has many generalizations to both dependence and unequal variances. In practice the convergence may be quite fast. If we for example want to compute the 1% quantile, the Central Limit Theorem approximation is often good enough with only 2 or 3 terms, if the parent distribution is not too skew, and all X_k 's are of the same order of magnitude. The level of approximation can easily be studied in a simulation study.

In the industrial example below we will consider life prediction for low cycle fatigue, where the damage driving parameter is the strain range $\Delta\epsilon$. The life model in this case is the Basquin-Coffin-Manson equation

$$\frac{\Delta\epsilon}{2} = \frac{\sigma_f'}{E} \cdot (2N)^b + \epsilon_f' \cdot (2N)^c \quad (6)$$

which in cases of large strain ranges can be simplified to the Basquin-Coffin-Manson equation

$$\frac{\Delta\epsilon}{2} \approx \epsilon_f' \cdot (2N)^c \quad (7)$$

which can be rewritten to the

$$N = e^a \cdot \Delta\epsilon^{1/c} \Leftrightarrow \ln N = a + \frac{1}{c} \ln \Delta\epsilon \quad \text{with} \quad a = -\ln 2 - \frac{1}{c} \ln(2\epsilon_f') \quad (8)$$

giving the life prediction model $\ln \hat{N} = f(\Psi, \hat{\theta}, \hat{\mathbf{X}})$ as a linear regression model, which will be used in our case study.

The logarithmic transformation used here has an important implication. Namely, the variation measures τ_i^2 and δ_j^2 can be interpreted as variation coefficients for the life, i.e.

$$\text{Var}[\ln N] \approx \frac{\text{Var}[N]}{E[N]^2}. \quad (9)$$

This interpretation is practical when one is forced to use engineering judgements for estimates of uncertainties, since they can easily be related to percentage uncertainty.

4. Estimation of prediction uncertainty

There are different kinds of uncertainties that need to be estimated. For scatter, the straight forward method is to make an experiment and calculate the sample standard deviation. In more complicated situations ANOVA (Analysis of Variance) is a useful tool. However, it is not always possible or economically motivated to perform experiments. Instead informed guesses, previous designs, or engineering experience

have to be used. Concerning statistical uncertainty, there are standard statistical methods like maximum likelihood theory for finding expressions of the variances of the estimates. In more complicated situations the bootstrap method can be applied. However, in order to use these statistical methods it is required that the original data is available, which is not always the case. A typical example is that only the estimated life curve is available, and maybe also information about the number of tests performed. In these situations it is also necessary to have a method in order to get an idea of the statistical uncertainty. For a specified model the model uncertainty is in fact a systematic error. However, if we consider the prediction situation such that we randomly choose a model from a population of models, then the systematic model error appears as a random error in the prediction. We will discuss different methods for estimating the model uncertainty, e.g. considering a random choice of models from the population of models, or considering extreme cases of models. In the following, we will give some more details and examples on the estimation, and motivate the estimated values in Table 1.

Life prediction	logarithmic life, ln(N)			
Type of scatter & uncertainty	scatter	uncertainty	total	comments
Strength scatter			0.38	
- Material, within shaft	0.15			See 4.1
- Material, between shafts	0.29			See 4.1
- Geometry	0.20			See 4.1
Statistical uncertainty			0.07	
- LCF-curve		0.07		See 4.2
Model uncertainty			0.84	
- LCF-curve		0.05		See 4.3
- Mean stress model		0.30		See 4.3
- Multi- to uni-axial		0.20		See 4.3
- Plasticity		0.72		See 4.3
- Stress analysis		0.24		See 4.3
- Temperature		0		See 4.3
Load scatter & uncertainty			0.50	
- Service load, scatter	0.40			See 4.4
- Service load, uncertainty		0.30		See 4.4
Total	0.55	0.90	1.05	

Table 1. Table summarizing the sources of scatter and uncertainty and their contributions, in terms of standard deviation of the logarithmic life, to the total prediction uncertainty.

4.1. Estimation of scatter

In our case study, the evaluation of the scatter is based on fatigue tests of three different shafts resulting in six observed lives each. From an ANOVA it was found that there is

both a within shaft scatter and a between shafts scatter, estimated at $\tau_{within} = 0.15$, and $\tau_{between} = 0.29$, respectively, see Figure 2. The within shaft scatter originates from material scatter, whereas the between shafts scatter is production scatter due to different batches, processing, or supplier effects.

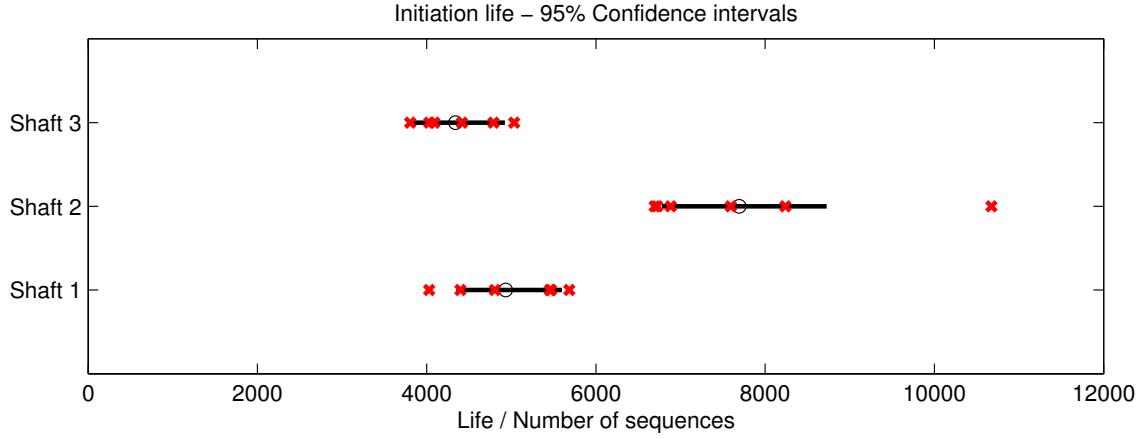


Figure 2. Confidence intervals for median life for the three different shafts tested.

In many situations it is necessary to rely on engineering judgement, and for example ask a man in the work shop “what is the worst case?”. The answer should often not be interpreted as representing zero probability of observing something more extreme, but rather that there is a very small risk of that. Therefore, the ‘worst case’ statement can be mathematically interpreted as a certain quantile, e.g. the 1/1000 risk of observing a more extreme case. By further assumptions, say a normal distribution, it is possible to estimate the scatter, τ , according to

$$z_{0.001} = \mu + 3\tau \Rightarrow \tau = (z_{0.001} - \mu) / 3 \quad (10)$$

where μ is the nominal value, $z_{0.001}$ is the ‘worst case’ representing the 0.1% quantile, and the value 3 comes from the 0.1% quantile of the standard normal distribution.

The strength of a structure and a component in the structure depends not only on the material properties, but is also highly dependent on geometry and assembling quality. For the example in Table 1, the geometry was varied according to the tolerances and the worst case resulted in a 10% change of the calculated stress. The worst case was here interpreted as the 0.1% quantile. Further, the sensitivity coefficient from stress to log-life was estimated at -6 , giving the estimated scatter, $\tau = 6 \cdot 0.10 / 3 = 0.20$.

4.2. Statistical uncertainty

There are standard statistical methods like maximum likelihood theory for finding expressions of the variances of estimates, see e.g. [Casella & Berger, 2001] or [Pawitan,

2001]. For more complicated estimation procedures the bootstrap method, which is a general method based on simulations, can be applied, see e.g. [Efron & Tibshirani, 1993], [Davison & Hinkley, 1997], and [Hjort, 1994].

In case of the Basquin-Coffin-Manson model (8), the parameters may be estimated using linear regression. In this situation the prediction uncertainty is

$$s \sqrt{1 + \frac{1}{n} + \frac{(x - \bar{x})^2}{\sum_i (x_i - \bar{x})^2}} \approx s \sqrt{1 + \frac{1}{n} + \frac{1}{n}} = s \sqrt{1 + \frac{2}{n}} \quad (11)$$

where x_i is the logarithmic strain during reference test i , x is the predicted logarithmic service strain, and the last expression is a rough approximation of the root expression. This simple approximation is correct if the squared distance from the actual value x to the reference test mean level \bar{x} is the same as the mean square distance in the reference tests. In case of interpolation use of the model this approximation is usually good enough.

The simplification (11) can be extended to models with more variables using the expression

$$s \sqrt{1 + \frac{r}{n}}, \quad (12)$$

where r is the number of parameters in the model. This kind of approximation is especially useful in cases where the original data is not available, but only the estimated life curve together with the observed experimental scatter and the number of tests performed. Thus, in these situations, a rough guess of the statistical uncertainty can be obtained as $\delta = s\sqrt{r/n}$. This was used for the case study, where a four-parameter Coffin-Manson curve from literature was used, which was based on 20 tests. Thus, the statistical uncertainty is estimated as $\delta = 0.15\sqrt{4/20} = 0.07$.

4.3. *Model uncertainty*

Assume that we have made life predictions based on different models, and want to address the uncertainty arising from the choice of model. We will consider two situations:

1. Assume that there is one model representing the least favourable case, and another representing the most favourable case. This means that these two models represent extreme cases of models, and all other models predict lives somewhere in between.
2. Assume that the models have been arbitrarily chosen; in other words they are randomly chosen from a population of models.

In the first situation, without any other information, it is natural to assume a uniform distribution with the end-points according to the extreme cases. This gives an estimate, based on the uniform distribution, of the standard deviation

$$s_U = \frac{\ln L_{\max} - \ln L_{\min}}{2\sqrt{3}} \quad (13)$$

with L_{\min} being the shortest predicted life, and L_{\max} the longest.

In the second situation, we may use the sample standard deviation as estimator

$$s = \sqrt{\frac{1}{n-1} \sum_{k=1}^n (\ln L_k - \overline{\ln L})^2}, \quad \overline{\ln L} = \frac{1}{n} \sum_{k=1}^n \ln L_k \quad (14)$$

where we have used n different life models predicting the lives L_1, L_2, \dots, L_n .

For the special case of two life predictions, we can compare the two approaches

$$s_U = \frac{1}{2\sqrt{3}} |\ln L_2 - \ln L_1| = 0.289 |\ln L_2 - \ln L_1|, \quad (15)$$

$$s = \frac{1}{\sqrt{2}} |\ln L_2 - \ln L_1| = 0.707 |\ln L_2 - \ln L_1|. \quad (16)$$

For very long load sequences (the normal situation for turbojet engine components) it is normal procedure to perform linear elastic finite element calculations. Plasticity will therefore have to be handled by plasticity models. For the plasticity model, the Linear rule and the Neuber rule for plastic correction can be seen as two extreme cases of models. In a similar situation the predictions using the two models differed by more than a factor two, and in that case the model uncertainty was estimated to be

$$\delta_{plasticity} = \frac{1}{2\sqrt{3}} (\ln 21 - \ln 1.77) = 0.72. \quad (17)$$

The model errors due to mean stress correction and the conversion from multi-axial to uniaxial stress state, were analyzed in the same manner. Further, the stresses were calculated using finite element programs and the uncertainty in the calculated stresses were judged to be about 4%. Thus, using the sensitivity coefficient of 6, the uncertainty in log-life is $\delta_{stress} = 6 \cdot 0.04 = 0.24$. Often in jet engine applications there are effects of high temperature. In this case this model error was judged to be negligible, hence $\delta_{temperature} = 0$.

Another approach to assess uncertainties in calculations is to rely on round robin studies. One example is when many different engineers have been given the same calculation task. It is then often the case that all engineers present different results. The engineers can be seen as randomly chosen from the population of all engineers, and the population scatter can be estimated. In such a way uncertainties due to calculations with different methods, programs and engineers can be judged, see e.g. [Pers et al., 1997] and [Bernauer et al., 2001].

4.4. Scatter and uncertainty in loads

The load that a component will experience during its time in service is usually very difficult to estimate. Experience from measurements in service for certain predefined manoeuvres gives a rough estimate of the expected load scenario, but variations and extreme events should also be considered.

Often this problem is hidden from the designer, since demands are already given by other departments in the company. However, it is important to estimate the load scatter and uncertainty, since this part of the load/strength problem may override other parts and make the variability in strength negligible.

In our example, the estimates are based on previous engineering experience. There is a scatter in the load due to the individual usage, which was judged to be $\tau_{load} = 0.40$. There is also an uncertainty whether the flight missions used for design really reflect the typical usage in field, where the uncertainty was judged to be $\delta_{load} = 0.30$.

4.5. Total prediction uncertainty

The total prediction uncertainty is the sum of all the contributions; see Eq. (4),

$$\delta_{pred} = \sqrt{\tau_1^2 + \tau_2^2 + \dots + \tau_p^2 + \delta_1^2 + \delta_2^2 + \dots + \delta_q^2} \quad (18)$$

which is the number in the right most bottom corner of Table 1. In this example it was reasonable to assume independence between the different sources of scatter and uncertainty, hence no covariance terms appear in Eq. (18). However, in case of correlation between the scatters and/or uncertainties, it is simply to add these covariance terms under the root sign in formula (18).

In the following section we will describe how the result can be used for life assessment. Another important use is to get information on where is it most efficient to try to reduce variation. In our case, the model uncertainty due to plasticity is by far the largest uncertainty, and thus it is motivated to further study the plasticity, in order to reduce its model uncertainty.

5. Reliability assessment

We will now demonstrate how the estimated total prediction uncertainty in log-life can be used and presented as a prediction interval for the life or as a safety factor for the life. The analysis is based on the construction of a prediction interval using a normal approximation

$$\ln N = \ln N_{pred} \pm z_p \cdot \delta_{pred} \Rightarrow N = N_{pred} \cdot \exp(\pm z_p \cdot \delta_{pred}) \quad (19)$$

where N_{pred} is the life prediction according to the calculation, δ_{pred} is our previously estimated total prediction uncertainty in log-life. The factor z_p is a quantile of the

standard normal distribution; $z_{0.025} = 2.0$ for a 95% interval, and $z_{0.001} = 3.0$ for a 99.8% interval.

In Figure 3 and Figure 4 the 95% prediction interval for the calculated life prediction according to Eq. (19), are compared to the ones obtained from fatigue life tests.

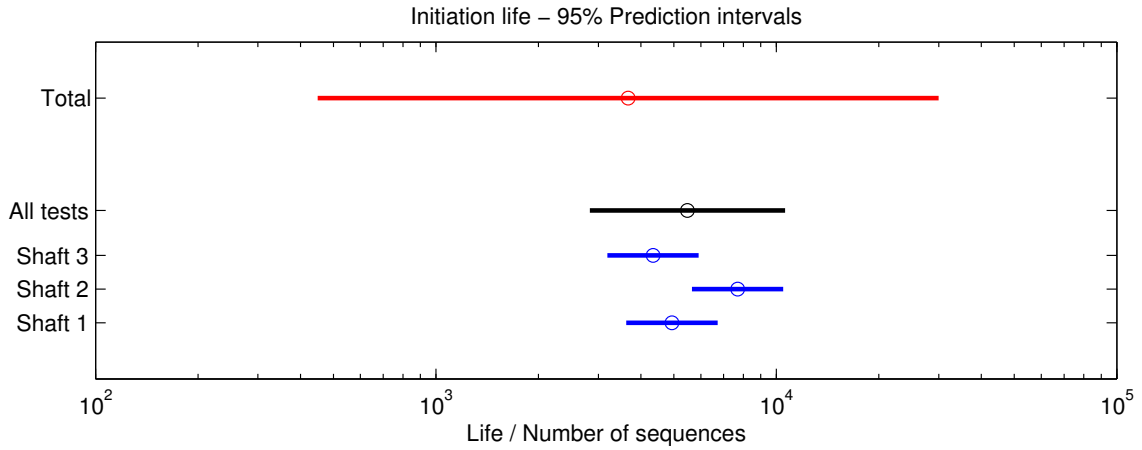


Figure 3. Prediction intervals for life, log-scale.

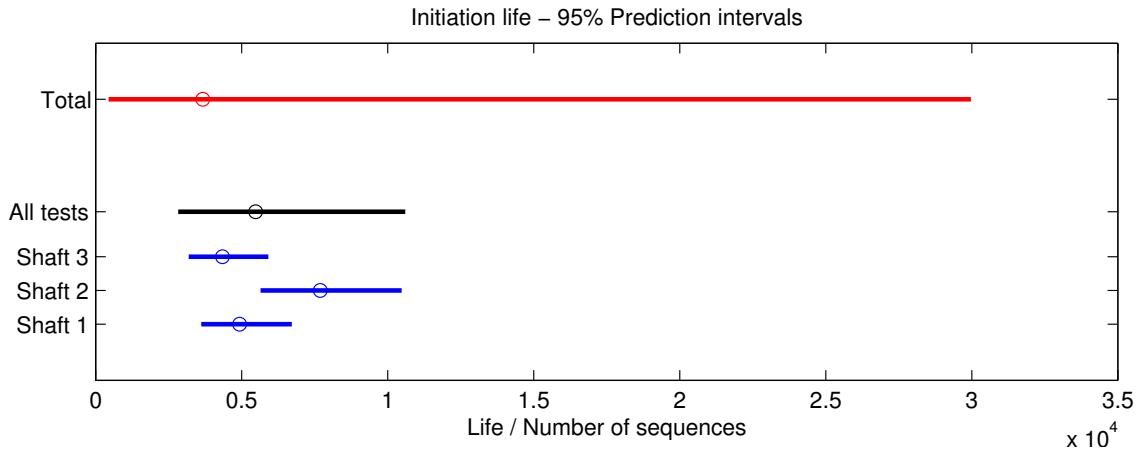


Figure 4. Prediction interval for life, linear scale.

The experiments revealed a scatter between the shafts. The experimental prediction intervals are therefore shown for each shaft individually as well as for the total scatter (within and between), compare Figure 2. Note that the experimental life interval can be seen as a lower limit of the length of the prediction intervals, since it involves only the scatter in life that is unavoidable. The experiments could be used to update the calculation of the life prediction, i.e. to correct for the systematic error and reduce some

of its uncertainties, especially the model uncertainties. This has not been pursued for this case study.

For design the lower endpoint of the prediction interval should be considered in order to get a reliable component. Often a reliability corresponding to a risk of 1/1000 is used. In the table below the limits of the 95% and 99.8% prediction intervals are presented.

Initiation life	Quantiles				
	0.1%	2.5%	50%	97.5%	99.9%
Life prediction	160	450	3 700	30 000	86 000

Table 2: Prediction quantiles for log-life.

It is possible to define a safety factor in life, based on the prediction interval, as the ratio between the median life and a low quantile of life

$$K_p = \frac{N_{0.5}}{N_p} \quad (20)$$

where p is the probability of failure. In our case it become

$$K_p = \exp(\ln N_{0.5} - \ln N_p) = \exp(z_p \cdot \delta_{pred}). \quad (21)$$

For the initiation life the safety factors in life becomes $K_{0.025} = 8.2$ and $K_{0.001} = 23$.

6. Updating the reliability calculation

If the total uncertainty in estimated life is dominated by uncertainties, i.e. the Z_j variables in (4), then it is desirable to arrange validation tests to reduce these uncertainties. Assume that a test with n individual test specimens has been performed. Each individual test result can be written as

$$\ln N_i = f(\Delta\epsilon_{tot,i}, \boldsymbol{\theta}) + \xi_i + z \quad (22)$$

where ξ_i is the individual test specific random error, which is the sum of the different random contributions, $\xi_i = \sum_j X_{ji}$, and z is the actual model error, which is the same for all individual tests and is denoted by a lower case letter to emphasize that it is non-random. However, it may be seen as an outcome of the random distribution of model errors.

The test result can be used for estimating the total model error:

$$\hat{z} = \overline{\ln N_{\bullet}} - f(\Delta\epsilon_{tot,\bullet}, \hat{\theta}) \quad (23)$$

i.e. by taking the average value of the differences between the observed lives and the predictions, where we have assumed that the random errors have zero mean.

The model can now be updated according to the experimental result, giving

$$\ln \hat{N} = f(\Delta\epsilon_{tot}, \hat{\theta}) + \hat{z}. \quad (24)$$

The prediction uncertainty for this updated model is now the sum of the variance of the scatter and the variance of the estimated model error \hat{z} . With reference to Eq. (23) this latter property can be found from

$$\text{Var}[\hat{z}] = \text{Var}[\overline{\ln N_{\bullet}}] + \text{Var}\left[f(\Delta\epsilon_{tot,\bullet}, \hat{\theta})\right]. \quad (25)$$

This is the sum originating from the statistical uncertainties in the original reference test and the new validation test. Thus, the first of these two terms comes from the statistical uncertainty in the model due to uncertain parameters, but will be decreased by averaging according to the number of individual test specimens

$$\delta_{\hat{z},\hat{\theta}}^2 = \frac{\delta_{\hat{\theta}}^2}{n}. \quad (26)$$

The second term comes from the random errors in the validation test and is estimated at

$$\delta_{\hat{z},\xi}^2 = \frac{s^2}{n} \quad (27)$$

where

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (\ln N_i - \overline{\ln N_{\bullet}})^2. \quad (28)$$

The uncertainty in the error estimate is

$$\delta_{\hat{z}}^2 = \delta_{\hat{z},\hat{\theta}}^2 + \delta_{\hat{z},\xi}^2. \quad (29)$$

6.1. *Uncertainty after updating*

In the previous section the uncertainty in model error estimate was investigated. If the validation tests are performed at conditions that is supposed to give the same model error as in service, then the uncertainty in the updated model is given by the random scatter in combination with the error uncertainty $\delta_{\hat{z}}$ above. If, in addition, the validation tests are performed on a random choice of components that represents all sources of scatter, then the total life uncertainty is

$$\text{Var}[\ln \hat{N}] = \delta_z^2 + \tau_\xi^2. \quad (30)$$

However, it will often be impossible to perform validation tests under true service conditions, and this fact may add new uncertainties to the predictions. Sources for such uncertainty components must be considered, and their estimated variance must be added to the variance above. This is done just in the same manner as described in the Section 4 and results in the prediction uncertainty after updating

$$\text{Var}[\ln \hat{N}] = \delta_z^2 + \tau_\xi^2 + \delta_{z'}^2, \quad (31)$$

where $\delta_{z'}^2$ is the variance of remaining model uncertainties, coming from future service conditions that were not represented in the validation test. For instance, the loads subjected to a component in a laboratory test will not be in full agreement with the multidimensional loads acting on the component in service. Uncertainties originating from this fact must be added in service prediction.

7. Conclusions and discussions

In the early design stages the basic and enhanced VMEA should be used, and based on the result of these, the most critical components are identified. For some components it may then be motivated to make a more sophisticated probabilistic VMEA, where the involved scatter and uncertainties need to be quantified. At first this can be made in quite a rough way, making use of the available information in the design process, but also making use of previous experience from similar structures, and other kinds of engineering experience. The goal is to get a rough estimate of the prediction uncertainty, and to locate the largest sources of scatter and uncertainty, in order to see where further efforts would be most efficient.

The proposed method represents the concept of First-Order Second-Moment (FOSM) reliability theory, see e.g. [Melcher, 1999] or [Ditlevsen & Madsen, 1996]. The First-Order refers to the linearization of the objective function, and the Second-Moment refers to the fact that only the means and variances are used.

In the present case study, the largest contribution to the prediction uncertainty originates from the model uncertainty due to the modelling of the plasticity. Therefore, in this specific case, it is motivated to further study the plasticity phenomenon. One may discuss whether the judgement of the uncertainty is realistic, and if the size of the contribution may be overestimated. However, it turns out that the model uncertainty is realistic, and it is motivated to model the plasticity phenomenon in more detail. With the computer performance of today, still increasing, it becomes more and more realistic to perform non-linear analyses. A load sequence containing some hundred load steps is today realistic to evaluate with a non-linear material model in the finite element calculation. The model uncertainty due to plasticity can then be expected to drop from 0.72 to 0.20, which would result in a reduction of the prediction uncertainty from 1.05 to 0.79.

Another example on the influence from tolerances was found on a similar component (also from a turbojet engine). The critical point of this component was a u-shaped

groove. The influence from tolerances on the fatigue life in the u-groove was entirely depending on the tolerances of the radius in the u-groove. A change of this radius, from minimum to maximum radius within the tolerance zone, resulted in a fourfold variation of fatigue life.

The split of the uncertainty sources in scatter and uncertainty gives possibilities to update the reliability calculation in a rational manner when new data are available. In case of experiments made under similar conditions as in service, the model errors may be estimated and corrected for. In case of measurements of service loads the corresponding uncertainty entry in Table 1 can be updated.

Acknowledgements

This work was supported in part by the Swedish Foundation for Strategic Research via the Gothenburg Mathematical Modelling Center (GMMC). The authors are thankful to Volvo Aero Corporation for collaboration and sharing data.

References

1. Bernauer G, Brocks W. Micro-mechanical modelling of ductile damage and tearing - results of a European numerical round robin, *Fatigue and Fracture of Engineering Materials and Structures*, Vol. 25, pp. 363-384, 2001.
2. Casella G, Berger R. *Statistical Inference*. (2nd Edition). Duxbury: California, 2001.
3. Chakhunashvili A, Barone S, Johansson P, Bergman B. Robust product development using variation mode and effect analysis. Submitted for publication.
4. Davis TP. Science, engineering, and statistics. *Applied Stochastic Models in Business and Industry*, Vol. 22, pp. 401–430, 2006.
5. Davison AC, Hinkley DV. *Bootstrap Methods and their Applications*. Cambridge UP: New York, 1997.
6. Ditlevsen O, Madsen H. *Structural Reliability Methods* (1st edition). John Wiley & Sons: Chichester, UK, 1996.
Internet edition 2.2.5, July 2005; <http://www.web.mek.dtu.dk/staff/od/books.htm> [25 April 2007].
7. Efron B, Tibshirani RJ. *An Introduction to the Bootstrap*. Chapman & Hall: New York, 1993.
8. Hjorth U. *Computer Intensive Statistical Methods: Validation Model Selection and Bootstrap*. Chapman & Hall: London, 1994.
9. Johansson P, Chakhunashvili A, Barone S, Bergman B. Variation Mode and Effect Analysis: a Practical Tool for Quality Improvement. *Quality and Reliability Engineering International*, Vol. 22, pp. 865-876, 2006

10. Lodeby K. *Variability Analysis in Engineering Computational Process*.
Licentiate thesis in Engineering, Mathematical Statistics, Chalmers, Göteborg,
2000.
11. Melchers R. *Structural Reliability Analysis and Prediction* (2nd edition). John
Wiley & Sons: Chichester, UK, 1999.
12. Pawitan Y. *In All Likelihood: Statistical Modelling and Inference Using
Likelihood*. Cambridge UP: New York, 2001.
13. Pers B-E, Kuoppa J, Holm D. Round robin test for finite element calculations,
*Proceedings of the First North European Engineering and Science Conference:
Welded High Strength Steel Structures* (Editor A. Blom), Stockholm, Sweden,
pp. 241-247, 1997.
14. Svensson T. Prediction uncertainties at variable amplitude fatigue.
International Journal of Fatigue, Vol. 19, pp. S295-S302, 1997.

Authors' biographies

Pär Johannesson received his MSc in Computer Science and Engineering at Lund Institute of Technology in 1993. Between 1994 and 1999 he was a PhD student at Mathematical Statistics, Lund Institute of Technology, and in 1999 he obtained his PhD in Engineering. During 2000-2001 he had a position as PostDoc at Mathematical Statistics, Chalmers University of Technology within a joint project with PSA Peugeot Citroën, where he spent year 2000 working for the Division of Automotive Research and Innovations at PSA in Paris. Currently he is working as an applied researcher at the Fraunhofer-Chalmers Research Centre for Industrial Mathematics in Göteborg.

Thomas Svensson received his PhD in Mathematical Statistics 1996 at Chalmers University of Technology. His main topic is statistical aspects on fatigue including modelling, test planning, and reliability. He has at present a position in fatigue research at the SP Technical Research Institute of Sweden.

Leif Samuelsson is working with method development in the fatigue area at Volvo Aero AB in Trollhättan. He received his MSc in Applied Mechanics at Chalmers University of Technology in 1972. He worked with structural analysis at Ericsson Radar Systems between 1972 and 1980. He proceeded with structural analysis on ships at the shipyard in Uddevalla from 1980 to 1982. In 1992 he moved to Volvo Aero, where a large part of the work was devoted to fatigue problems. He worked at SAAB Automobile from 1987 to 1992, mainly with load analysis and fatigue. In 1992 he moved back to Volvo Aero.

Jacques de Maré got his PhD in mathematical statistics in 1975 from Lund University. He worked at Umeå University from 1976 to 1979 before securing a position at Chalmers University of Technology. He became professor there in 1995. He was a visiting researcher at the University of North Carolina in 1982, at the University of

California, Santa Barbara, in 1989 and at Kyushu University in Fukuoka 2005. He is a member of the International Statistical Institute and serves as a scientific advisor at the Fraunhofer-Chalmers Centre. He is currently working with statistical methods for material fatigue in co-operation with SP Technical Research Institute of Sweden. At Chalmers he has also worked in different ways to bring mathematical and engineering disciplines closer together.

Bo Bergman has been SKF professor of Total Quality Management at Chalmers University of Technology since 1999. Earlier, from 1984 to 1999, he was a professor of Quality Technology and Management at the University of Linköping. From 1969 to 1984, he was employed by Saab Aerospace in different engineering and managerial positions in the areas of reliability, quality and statistics. During that period he gained his PhD in Mathematical Statistics at Lund University and at the end of that period was also an adjunct professor in Reliability Engineering at the Royal Institute of Technology in Stockholm. He is a member of ASQ and the Scandinavian Society of Reliability Engineers and an elected member of the International Statistics Institute. He is an academician of the International Academy for Quality