CENTRAL LIMIT THEOREM PROOF

1 Central Limit Theorem

Let $X_1, ..., X_n$ be independent and identically distributed random variables with mean μ and variance σ^2, which is assumed to be finite. Let $\bar{X} = (X_1 + ... + X_n)/n$, and define

$$W_n = \frac{\bar{X} = -\mu}{\sigma/\sqrt{n}}.$$

The distribution of W_n converges to a standard normal distribution as $n \to \infty$.

2 Proof

The central limit theorem can be proved using

1. Moment Generating Functions
2. Taylor Series Expansions

Let $Y_i = (X_i - \mu)/\sigma$, so Y_i has mean zero and variance one. The moment generating function of W_n can be written as

$$M_W(t) = E\left(e^{tW}\right) = \left[M_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n.$$

where M_Y is the moment generating function of Y. If we use a Taylor Series to expand $M_W(t)$ about zero, we have

$$\begin{aligned} M_W(t) &= \left[M_W(0) + M_W'(0)\frac{t}{\sqrt{n}} + \frac{1}{2}M_W''(0)\frac{t^2}{n} + \cdots\right]^n \\ &= \left[1 + \frac{t^2}{2n} + \cdots\right]^n \end{aligned}$$

We can rewrite this moment generating function as

$$M_W(t) = exp\left\{n \ln\left(1 + \frac{t^2}{2n} + \cdots\right)\right\}.$$

Now expand $\ln(1+x)$ in a Taylor Series in the expression above to obtain

$$M_W(t) = exp\left\{n\left(\frac{t^2}{2n} + \cdots\right)\right\} \; = \; exp\{t^2/2\}$$

which happens to be the moment generating function of a standard normal random variable.